Due Wed

then 2nd 1st

$A\boxed{B\vec{x}}$

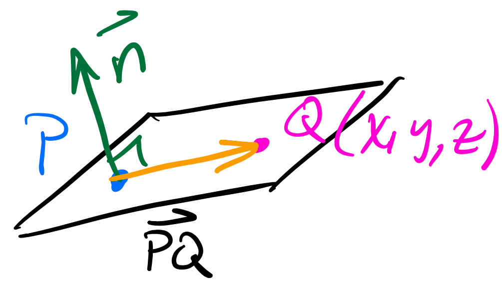
$A R_0 \vec{x}$

3.3 – Orthogonality

Definition 1: Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to every vector in R^n.

A vector $\mathbf{n}$ that is orthogonal to a line in R^2 or R^3 or a plane in R^3 is called a **normal**.

3. Find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal.
$P(-1, 2, -1)$, $\mathbf{n} = (-2, 1, -1)$



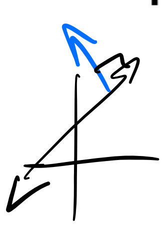
Let Q(x, y, z) be any other point in the plane.

$$\vec{n} \cdot \vec{PQ} = 0 \Rightarrow (-2, 1, -1) \cdot (x+1, y-2, z+1) = 0$$

$$\Rightarrow -2(x+1) + (y-2) - (z+1) = 0$$

$$\boxed{-2x + y - z = 5}$$

Theorem 3.3.1



a) If a and b are constants that are not both zero, then an equation of the form
$ax + by + c = 0$ represents a line in R^2 with normal $\mathbf{n} = (a, b)$.

b) If a, b, and c are constants that are not all zero, then an equation of the form
$ax + by + cz + d = 0$ represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

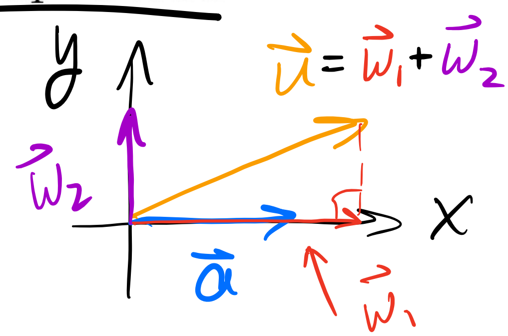
Theorem 3.4.3 (not a typo)
If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.

Consider the homogeneous system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Shifting gears

Theorem 3.3.2 Projection Theorem
If $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^n, and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.



$$\vec{w}_1 = k\vec{a} \Rightarrow \vec{u}\cdot\vec{a} = (\vec{w}_1 + \vec{w}_2)\cdot\vec{a}$$
$$= (k\vec{a} + \vec{w}_2)\cdot\vec{a}$$
$$= k\,\vec{a}\cdot\vec{a} = k\|\vec{a}\|^2$$

$$k = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2} \Rightarrow \vec{w}_1 = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}$$

When $\mathbf{u}$, a vector in R^n, is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of a vector $\mathbf{a}$ in R^n, $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **orthogonal projection of u on a** or the **vector component of u along a**. The vector $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **vector component of u orthogonal to a**.

$\vec{w}_1$

19. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$. $\vec{w}_2$

$\mathbf{u} = (2, 1, 1, 2)$, $\mathbf{a} = (4, -4, 2, -2)$

$$\vec{w}_1 = \text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} a = \frac{8-4+2-4}{16+16+4+4} \vec{a}$$

$$= \frac{2}{40} \vec{a} = \frac{1}{20}(4, -4, 2, -2)$$

$$\vec{w}_1 = \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10}\right)$$

$$\underline{\vec{u} = \vec{w}_1 + \vec{w}_2} \Rightarrow \vec{w}_2 = (2, 1, 1, 2) - \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10}\right)$$

$$\vec{w}_2 = \left(\frac{9}{5}, \frac{6}{5}, \frac{9}{10}, \frac{21}{10}\right)$$

Checks: $\vec{w}_1 + \vec{w}_2 = (2, 1, 1, 2) = \vec{u}$ ✓

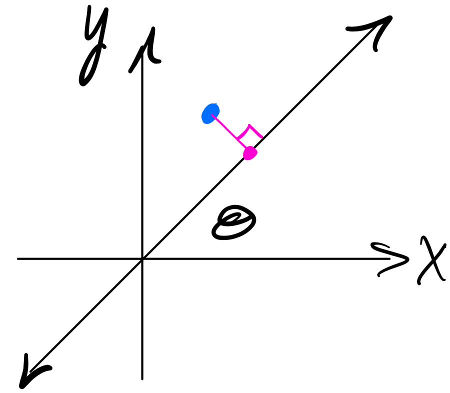
$$\vec{w}_2 \cdot \vec{a} = \frac{36}{5} - \frac{24}{5} + \frac{9}{5} - \frac{21}{5} = 0$$ ✓

38. Find the standard matrix for the orthogonal projection of R^2 onto the stated line, and then use that matrix to find the orthogonal projection of the given point onto that line.

The orthogonal projection of $(1, 2)$ onto the line that makes an angle of $\pi/4$ $(= 45°)$ with the positive x-axis.

$$A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \right]$$

Find the images of $\vec{e}_1, \vec{e}_2$ under T_A.



$$\vec{a} = (\cos\theta, \sin\theta)$$

$$\operatorname{proj}_{\vec{a}} \vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \cos\theta(\cos\theta, \sin\theta)$$
$$= (\cos^2\theta, \sin\theta\cos\theta)$$

Likewise, $\operatorname{proj}_{\vec{a}} \vec{e}_2 = \sin\theta(\cos\theta, \sin\theta)$
$$= (\sin\theta\cos\theta, \sin^2\theta)$$

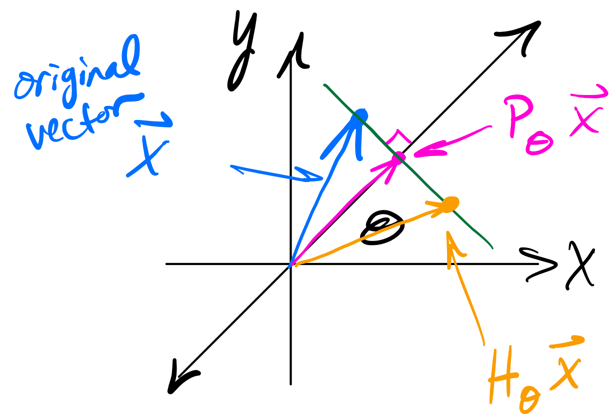
$$P_\theta = [T_A] = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$

$$P_{\pi/4} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P_{\pi/4}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

36. Find the standard matrix for the reflection of R^2 about the stated line, and then use that matrix to find the reflection of the given point about that line.

The reflection of $(1,2)$ about the line that makes an angle of $\pi/4$ $(=45°)$ with the positive x-axis.



The distance between $\vec{x}$ and $P_\theta \vec{x}$ is half the distance between $\vec{x}$ and $H_\theta \vec{x}$

$$P_\theta \vec{x} - \vec{x} = \frac{1}{2}\left(H_\theta \vec{x} - \vec{x}\right)$$

$$\Rightarrow H_\theta \vec{x} = 2P_\theta \vec{x} - \vec{x} = (2P_\theta - I)\vec{x}$$

$$\Rightarrow H_\theta = 2P_\theta - I = \begin{bmatrix} 2\cos^2\theta - 1 & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$H_{\pi/4} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow H_{\pi/4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Theorem 3.3.3 Theorem of Pythagoras in R^n

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in R^n with the Euclidean inner product, then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Pf: $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

$$= \vec{u}\cdot\vec{u} + 2\underbrace{\vec{u}\cdot\vec{v}}_{=0} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 \checkmark$$

$$\begin{array}{ccc} \vec{u} & & \vec{v} \\ T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, & & T\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 7 \\ 8 \end{bmatrix} \end{array}$$

$$\vec{e}_1 = a\vec{u} + b\vec{v} \Rightarrow T(\vec{e}_1) = \underline{a}\,T(\vec{u}) + \underline{b}\,T(\vec{v})$$

$$\vec{e}_2$$

$$\left[\begin{array}{cc|cc} \vec{u} & \vec{v} & 1 & 0 \\ & & 0 & 1 \end{array}\right]$$

$$T_A(\vec{x}) = A\vec{x} \qquad \vec{e}_3$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \\ i \end{bmatrix}$$

$$\begin{array}{ccc} T_A(\vec{e}_1) & T_A(\vec{e}_2) & T_A(\vec{e}_3) \end{array}$$

$$T(k\vec{u}+\vec{v}) = kT(\vec{u})+T(\vec{v})$$